Bias and Division in the Free World

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Given $\sigma^2 \in (0,\infty)$, a theorem of Kolmogorov states that $X \in \mathbb{ID}_{0,\sigma^2}$, the set of all infinitely divisible random variables with mean zero and variance $\sigma^2 \in (0,\infty)$, if and only if there exists a probability measure ν on $\mathbb{R}$ such that the characteristic function φ of X satisfies

$$\phi(t) = \exp\left(-\frac{\sigma^2 t^2}{2}\nu(\{0\})\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} \frac{e^{itx}-1-itx}{x^2}\,\nu(dx)\right), \quad t \in \mathbb{R}. \quad (1)$$

From Stein's method, for every mean zero, variance σ^2 random variable X there exists a unique 'X-zero bias' distribution $\mathcal{L}(X^*)$ such that

$$E[Xf(X)] = \sigma^2 E[f'(X^*)] \quad \text{for all Lipschitz}_1 \text{ functions } f.$$

The mapping $\mathcal{L}(X) \to \mathcal{L}(X^*)$ has the Gaussian $\mathcal{N}(0,\sigma^2)$ distribution as its unique fixed point. Using probabilistic techniques, we show that $X \in \mathbb{ID}_{0,\sigma^2}$ if and only if

$$X^* =_d X + UY$$

where $=_d$ denotes equality in distribution, with X, U, Y independent and $U \sim \mathcal{U}[0,1]$,

Similarly, in free probability, we show that for all mean zero, variance $\sigma^2 \in (0,\infty)$ random variables there exists a unique distribution X° such that

$$E[Xf(X)] = \sigma^2 E[f'(UX^\circ + (1-U)Y^\circ)] \quad \text{for all Lipschitz}_1 \text{ functions } f,$$

where $Y^\circ =_d X^\circ$, the variables X°, Y°, U are independent, and $U \sim \mathcal{U}[0,1]$. The mapping $\mathcal{L}(X) \to \mathcal{L}(X^\circ)$ has the $\mathcal{S}(0,\sigma^2)$ semi-circle distribution as its unique fixed point, and $X \in \mathbb{FID}_{0,\sigma^2}$, the set of all freely infinitely divisible random variables with mean zero and variance $\sigma^2 \in (0,\infty)$, if and only if there exists a random variable Y such that, with G_W denoting the Cauchy transform of W,

$$G_{X^\circ}(z) = G_{Y^\sharp}(1/G_X(z)) \quad \text{where} \quad G_{Y^\sharp}(z) = \sqrt{G_Y(z)/z}.$$

These new identities lead to probabilistic interpretations of the corresponding Lévy measures, such as ν in (1) in the classic case.